B.Sc. Part I (Hons.)

1st Paper

Rank of a matrix (continued)

Some important points

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□ Elementary transformation of a matrix

- interchange of any two rows/two columns of a matrix

C_{ij} or R_{ij}

- multiplication of all elements of any row/column by a non-zero number

KC_i or $C_i(K)$
KR_i or $R_i(K)$

- addition of all elements of any row/column of a matrix, the same multiple of the corresponding elements of any other row or columns

$R_i + KR_j$ or $R_{ij}(K)$

stands for addition of K-times the jth row of a matrix to the ith row

Similarly $C_i + KC_j$ or $C_{ij}(K)$

* Equivalent matrices

If a matrix B is obtained from A by a finite no. of elementary transformations, B is said to be equivalent to A. Denoted by

$B \sim A$.

Also, (a) $A \sim A$,

(b) $B \sim A \Rightarrow A \sim B$

(c) $A \sim B,\ B \sim C \Rightarrow A \sim C$

* Elementary matrices

A matrix obtained from a unit matrix by some elementary row (or column) transformations is called an elementary row (or column) matrix. It is denoted by E-matrix.

* All elementary matrices are non-singular.

* Rank of a matrix does not change after elementary transformations.

* Normal form of a matrix

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_n & 0 \end{bmatrix}, \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \begin{bmatrix} I_n \end{bmatrix}$$

where I_n is the n-rowed unit matrix.

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ First canonical form

* Every non-zero matrix of rank r can be reduced to any of the normal forms.

Example 1. Find the rank of

$$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}.$$

Solution

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{bmatrix} \quad C_2 \to C_2 + C_1,\ C_3 \to C_3 - 3C_1,\ C_4 \to C_4 - 6C_1$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix} \quad R_2 \to R_2 - R_1,\ R_3 \to R_3 - 5R_1$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 10 \\ 0 & 2 & 2 & 19 \end{bmatrix} \quad C_2 \to \frac{1}{4} C_2,\ C_3 \to -\frac{1}{6} C_3,\ C_4 \to -C_4$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad C_3 \to C_3 - C_2,\ C_4 \to C_4 - 10C_2$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad C_4 \leftrightarrow C_3$$

[PTO]

$\Rightarrow$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_3 \to (-1) \times C_3$$

$$\Rightarrow A \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

So, the rank of A is $\underline{3}$.

Q.2 Find the rank of $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Soln

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix} \begin{array}{l} C_2 \to C_2 + C_1, \\ C_3 \to C_3 + 2C_1, \\ C_4 \to C_4 + 4C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 1 & 7 \\ 3 & 4 & 3 & 10 \\ 6 & 9 & 4 & 17 \end{bmatrix} C_3 \to \frac{1}{3} C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 7 \\ 0 & 4 & 3 & 10 \\ 0 & 9 & 4 & 17 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_6 \to R_6 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 4 & 10 \\ 0 & 4 & 9 & 17 \end{bmatrix} C_3 \leftrightarrow C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -11 & -11 \\ 0 & 0 & -11 & -11 \end{bmatrix}$$

$R_4 \to R_4 - 4R_1 \quad R_3 \to R_3 - 3R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 \times (-\frac{1}{11}) \\ R_4 \to R_4 \times (-\frac{1}{11}) \end{matrix}$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 - 5C_2 \\ C_4 \to C_4 - 7C_2 \end{matrix}$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_4 \to C_4 - C_3$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank of $A = \underline{\underline{3}}$

$\boxed{Q.}$ Find the rank of $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, by reducing it to normal form.

Soln.

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

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$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$$

$C_3 \to C_3 - C_1$
$C_4 \to C_4 - C_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1, \; R_4 \to R_4 - R_1$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad C_3 \to -\frac{1}{3} C_3, \; C_4 \to (-1) \times C_4$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2, \; C_4 \to C_4 - C_2$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2, \; R_4 \to R_4 - R_2$$

$$\Rightarrow A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 2$.